Application of Moment Expansion Method to Options Square Root Model

MidYear Progress Report for AMSC 663

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Abstract

The Options Square Root Model or Heston Model is the stochastic volatility model developed by Heston (1993). The governing equations consider not only the stochastic spot return but also stochastic volatility, which has a correlation with spot return. Heston (1993) gave a closed-form solution for the European Call option price based on Fourier Transform method. In this project, we implement a moment expansion method to solve the Options Square Model and the solution is compared with the Fourier Transform based solution. The moment generating function is used to derive from first to at least sixth order moments to calculate the options price. This moment expansion based solution is compared with Fourier Transform based solution in terms of accuracy and implementation difficulty.

1 Background

Because it is easy to calculate and explicitly model the relationship of all the variables, the Black-Scholes Model has been widely and successfully used in explaining stock option prices. However, the strong assumption in Black-Scholes Model that the stock returns are normally distributed with constant variance and mean is not true in reality. Emprical study shows that in reality security prices do not follow a strict stationary log-normal process and the variance is not a constant. Starting from this point, Hull and White (1987) proposed a new model with stochastic volatility. However, these types of models could not provide a closed form solutions and involve more numerical techniques. Heston(1993) proposed a new stochastic volatility model describing the evolution of volatility of the underlying asset. He also provided a closed-form solution. The basic Heston model assumes that S_t, the price of the asset, is determined by a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^s \quad (1)$$

where ν_t, the instantaneous variance, is a CIR (Cox-Ingersoll-Ross) process:

$$dv_t = k(\theta - v_t)dt + \sigma \sqrt{v_t} dW_t^v \quad (2)$$

$$dW_t^s dW_t^v = \rho dt$$

and dW_t^s, dW_t^v are Wiener Processes with correlation ρ.

The parameters in the above equations represent the following:

- μ is the average rate of return of the asset.
- θ is the long vol, or long run average price volatility; as t tends to infinity, the expected value of ν_t tends to θ.
- κ is the rate at which ν_t reverts to θ.
- σ is the vol of vol, or volatility of the volatility, i.e, the variance of ν_t.

The Wiener Process W_t is characterized by three facts:

1. $W_0 = 0$
2. W_t is almost surely continuous

3. W_t has independent increments with distribution $W_t - W_s \sim N(0, t-s)$ (for $0 \le s < t$).

$N(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2. The condition that it has independent increments means that if $0 \le s_1 \le t_1 \le s_2 \le t_2$ then $W_{t1} - W_{s1}$ and $W_{t2} - W_{s2}$ are independent random variables, and the similar condition holds for n increments. An alternative characterization of the Wiener Process is an almost surely continuous martingale with $W_0 = 0$ and quadratic variation $[W_t, W_t] = t$ (which means that W_t^2-t is also a martingale).

The CIR process is a Markov process with continuous paths defined by the following stochastic differential equation:

$$dr_t = -\theta(r_t - \mu)dt + \sigma\sqrt{r_t}dW_t$$

where θ and σ are parameters. Value r_t follows a noncentral Chi-Square distribution. The CIR process is widely used to model short term interest rate.

2 Approach

Let $x = \ln S(t)$, the spot return, according to equation (1), we can have

$$dx = (\mu - \tfrac{1}{2}v)dt + \sqrt{v}dW_t^s$$

with terminal condition at the expiration time $M(x, v, 0, n) = x^n$.

Then, with equation (2) $dv = k(\theta - v)dt + \sigma\sqrt{v}dW_t^v$ to formulate the Kolmogorov Backward Equation:

$$\tfrac{1}{2}vM_{xx} + \rho\xi vM_{xv} + \tfrac{1}{2}\xi^2 vM_{vv} + (\mu - \tfrac{1}{2}v)M_x + k(\theta - v)M_v = M_\tau \tag{3}$$

Guess $M(x, v, \tau, n) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} C_{ij}{}^{n}(\tau)x^i v^j$, with initial conditions $C_{n0}(0) = 1, C_{ij}(0) = 0$. τ here is the time to expiration time, i.e., $T - t$, when T is the maturity time and t is the current time.

Considering n=1, then

$$M(x, v, \tau, 1) = C_{10}(\tau)x + C_{01}(\tau)v + C_{00}(\tau) \tag{4}$$

Substitute equation (4) into (3), we have

$$(\mu - \tfrac{1}{2}v)C_{10}(\tau) + k(\theta - v)C_{01}(\tau) = C_{10}^{'}(\tau)x + C_{01}^{'}(\tau)v + C_{00}^{'}(\tau)$$

Then we can get two ordinary differential equations,

$$-(\tfrac{1}{2}C_{10}(\tau) + kC_{01}(\tau))v = C_{01}^{'}(\tau)v \tag{5}$$

$$\mu C_{10}(\tau) + k\theta C_{01}(\tau) = C_{00}^{'}(\tau) \tag{6}$$

Solve equation (5) and (6) with initial conditions, we get

$$C_{01} = \frac{-0.5 + 0.5e^{-k\tau}}{k}, C_{00} = \mu\tau + \frac{(0.5 - 0.5e^{-k\tau})\theta}{k} - 0.5\tau\theta$$

Then the 1st moment $M(x, v, \tau, 1) = \mu\tau + \frac{(0.5 - 0.5e^{-k\tau})\theta}{k} - 0.5\tau\theta + \frac{(-0.5 + 0.5e^{-k\tau})v}{k} + x$.

Similarly, using backward equation (3) will generate a group of linear ordinary differential equations, moments could be got after solving these ordinary differential equations. The results up to 3rd moments and the derivation process are in the Appendix.

Next, we need to consider the way to implement moments into option price formula. European Call Option payoff at the expiration time T is the maximum of 0 and the stock price at the expiration time minus the strike price, i.e., $C(T,S(T))=(0,S(T)-K)^+$. The similar method will be used in this study as Corrado and Su (1996). They used truncated Gram-Charlier series expansion of the density function up to 4th moment. The resulting truncated density gave an estimation of the nonnormal skewness and kurtosis as following:

$$g(z)=n(z)\left[1+\tfrac{\mu_3}{3!}(z^3-3z)+\tfrac{\mu_4-3}{4!}(z^4-6z^2+3)\right] \quad (7)$$

where $n(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}$, $z=\dfrac{\ln(S_t/S_0)-(\mu-\sigma^2/2)t}{\sigma\sqrt{t}}$, $\mu_{3,}\mu_4$ are the 3rd and 4th moment respectively, σ is the standard deviation of returns for the underlying stock, which is a constant.

If z is standard normal, then the skewness and kurtosis are $\mu_3=0, \mu_4=3$, which reduced the truncated density function (7) as the standard normal density function.

Based on this truncated Gram-Charlier density expansion, they give option price formula as following:

$$C_{GC}=C_{BS}+\mu_3 Q_3+\mu_4 Q_4 \quad (8)$$

where
$$Q_3=\tfrac{1}{3!}S_0\sigma\sqrt{t}((2\sigma\sqrt{t}-d)n(d)-\sigma^2 tN(d))$$
$$Q_4=\tfrac{1}{4!}S_0\sigma\sqrt{t}((d^2-1-3\sigma\sqrt{t}(d-\sigma\sqrt{t}))n(d)+\sigma^3 t^{3/2}N(d))$$
$C_{BS}=S_0N(d)-Ke^{-rt}N(d-\sigma\sqrt{t})$ is the Black-Scholes option pricing formula
$$d=\frac{\ln(S_0/K)+(r+\sigma^2/2)t}{\sigma\sqrt{t}}.$$

After obtaining the 3rd and 4th moments using the moment expansion method, we can implement these moments into equation (8) as $\mu_{3,}\mu_4$, with the same Q_3,Q_4 part. Then the call option price can be calculated.

To determine which nth moment is enough to satisfy accuracy requirement, we will use the Fourier transform solution as the truth and compare the relative error. Therefore, we can use the moment expansion solutions as an approximation of the Fourier transform solution.

Heston (1993) guessed a solution to the Heston model, which involves two parts, one is the present value of the spot asset before optimal exercise, and the other is the present value of the strike-price payment. The solution has the following form:

$$C(s,v,t)=SP_1-KP(t,T)P_2$$

Where $P(t,T)=\mathrm{e}^{-(T-t)}$ is the price at time t of a unit discount bond that matures at time T. Both of these two terms satisfy equation (3).

$P_{1,}P_2$ satisfy the terminal condition,

$$P_j(x,v,T;\ln[K])=1_{(x\geq \ln[K])}.$$

and have characteristic functions $f_j(x,v,t;\phi)$ respectively which also satisfy equation (3). $P_{1,}P_2$ can be obtained from the solutions of these characteristic functions. Then the call option prices can be obtained.
To check the accuracy, we compare the Fourier Transform based solutions C_F with 1st to nth order moment expansion based solutions C_M^n. The graph of $\| C_M^n - C_F \|$ and n could help us to determine a cutoff and find the good enough estimation.

3 Testing

I programmed in Matlab the exact solutions given by Heston (1993) for comparison. The test is using the same parameters as Table 1.
Fig.1 shows that the effect of negative and positive correlation of the two Wiener Process in option price. When the correlation is positive, option price with Heston model has more value than the results with Black-Scholes model when call option is out of the money, while less value than the results with Black-Scholes model when call option is in the money. The negative correlation has the opposite effect. Fig.2 shows the effect of increasing of volatility of volatility. Increasing volatility makes the option price with Heston model has more value when the stock price is far from the strike price and less value when the stock price is near the strike price. All these results mathes Heston Paper.
Fig.3 is the marginal effect of nonnormal skewness and kurtosis, i.e., Q_3, Q_4 part in equation (8). -Q_3 has similar effect with correlation in Fig.1 while Q_4 has similar effect with volatility of volatility. These two parts will not change when moments from moment expansion method are applied to equation (8).

$$Moneyness(\%) = \frac{Ke^{-rt} - S_0}{Ke^{-rt}} \times 100$$

Parameters	Value in Fig.1	Value in Fig. 2	Value in Fig.3
Mean reversion	2	2	-
Long-run variance	0.01	0.01	0.15
Initial variance	0.01	0.01	-
Correlation	0.5/-0.5	0	-
Volatility of volatility parameter	0.1	0.1/0.2	-
Option maturity	0.5 year	0.5 year	0.25
Interest rate	0	0	0.04
Strike Price	100	100	75-125 (S0=100)

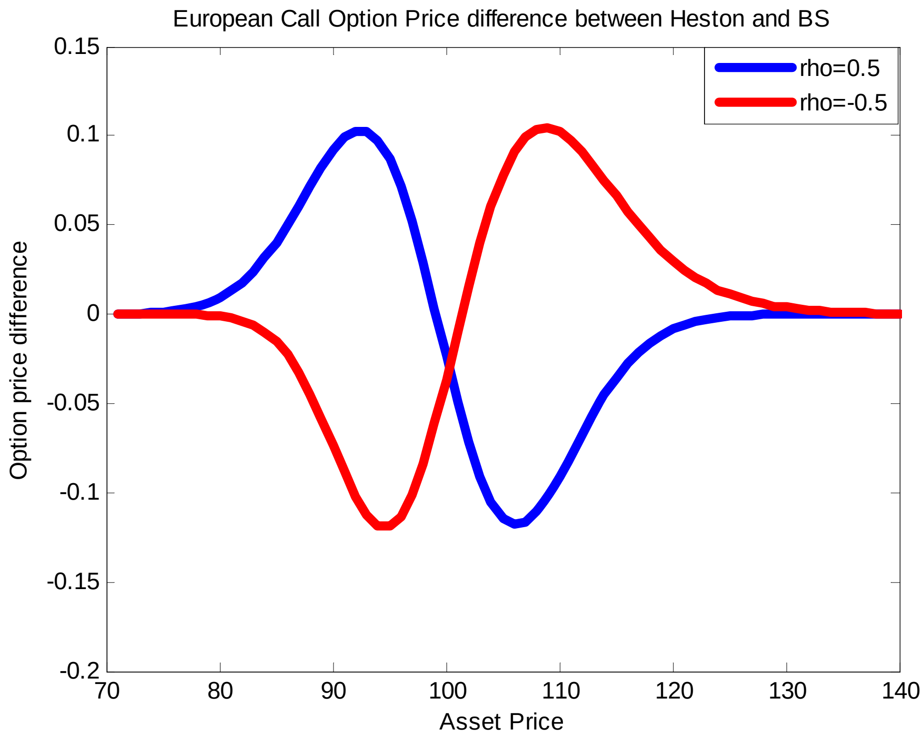

Fig. 1 Option price with Heston Model minus Price with Black Scholes Model on different correlation

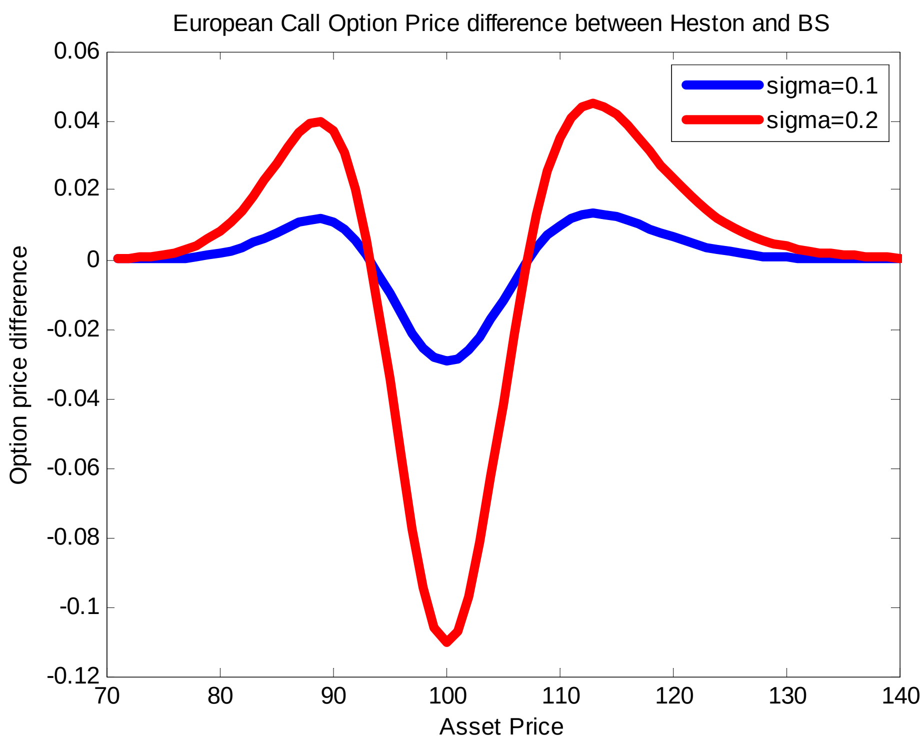

Fig. 2 Option price with Heston Model minus Price with Black Scholes Model on different vol. of vol.

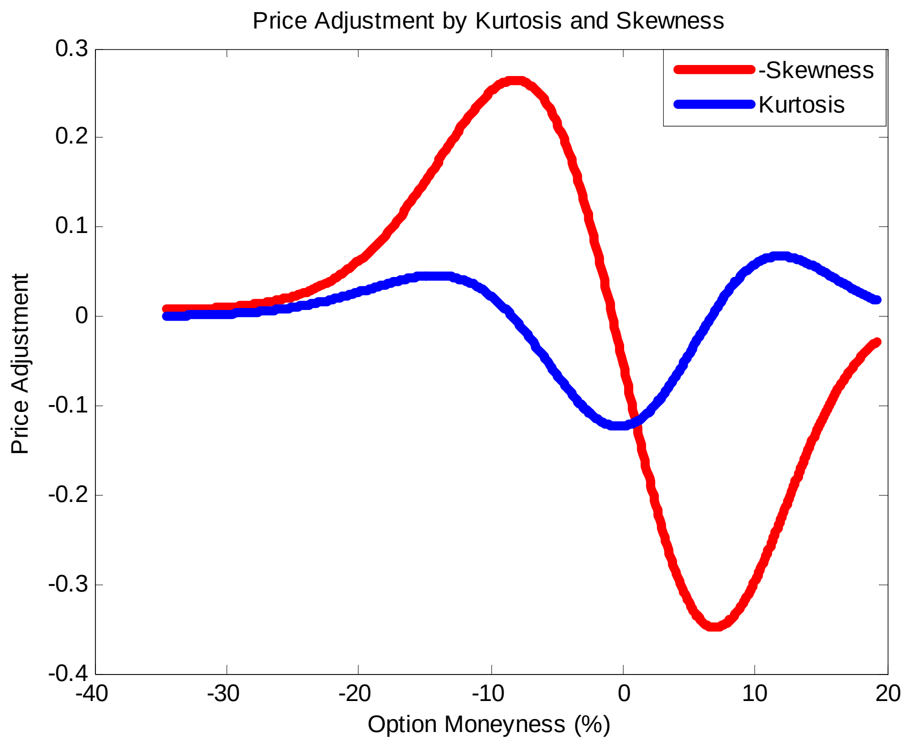

Fig. 3 Adjustment part (-Q3) and Q4 in equation (8)

4 Future Work

1) Continue working on solving group of linear ODEs numerically in Matlab for any nth order. Currently, the code is debugging for this. To solve this, the iterative way is to generate the ODEs (see Appendix) is used. We can either solve these ODEs iteratively, or use matrix exponential method which requires finding the coefficient matrix first.
2) Implement the solved moments into the equation (8) to get the European Call Option formula based on the Gram-Charlier density expansion.
3) Test and validate the Fast Fourier Transfer (FFT) method for Heston Model. The code which is designed for general stochastic model is obtained from Jun Wang (AMSC). I need to implement the characteristic function for Heston model into the code. Comparing the testing results with the exact solutions to Heston model (present in Part 3) can be used for validation.
4) Compare the computed option price by moment expansion method with the FFT solutions and the exact solutions for Heston model to determine the cutoff order of moment that need to use for option price estimation.

5 References

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6 Appendix: Derivation of Moments

Starting from the Backward Equation:

$$\tfrac{1}{2}vM_{xx} + \rho\xi vM_{xv} + \tfrac{1}{2}\xi^2 vM_{vv} + (\mu - \tfrac{1}{2}v)M_x + k(\theta - v)M_v = M_\tau \tag{3}$$

we can generate a group of linear ordinary differential equations as following,

$$C'_{i,j} = kjC_{i,j} + \tfrac{1}{2}(i+2)(j+1)C_{i+2,j-1} + (\rho\sigma(i+1)j + \mu(i+1))C_{i+1,j} + (\tfrac{1}{2}\sigma^2(j+1)j + k\theta(j+1))C_{i,j+1} - \tfrac{1}{2}(i+1)C_{i+1,j-1} \tag{9}$$

with initial conditions $C_{n0}(0) = 1, C_{ij}(0) = 0$.

One method is to solve these ODEs interatively, the other one is to use matrix exponential.

$$y'(t) = A\,y(t)$$

Then the solution is $y(t) = e^{At}\,b$, with initial value b.

Find out the eigenvalues λ_j and eigenvectors z_j of matrix A, the solution can be written into

$$y(t) = \sum_{j=1}^{n} b_j e^{\lambda_j t} z_j \tag{10}$$

The coefficient matrix A:

n=1

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.5 & -k \\ 0 & \mu & k\theta \end{pmatrix} \text{ with corresponding } y(t) = \begin{pmatrix} C_{10} \\ C_{01} \\ C_{00} \end{pmatrix}$$

n=2

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -k & 0 & 0 & 0 & 0 \\ 0 & -0.5 & -2k & 0 & 0 & 0 \\ 2\mu & k\theta & 0 & 0 & 0 & 0 \\ 1 & \mu+\rho\sigma & \sigma^2+2k\theta & -0.5 & -k & 0 \\ 0 & 0 & 0 & \mu & k\theta & 0 \end{pmatrix} \quad \text{with corresponding } y(t)=\begin{pmatrix} C_{20} \\ C_{11} \\ C_{02} \\ C_{10} \\ C_{01} \\ C_{00} \end{pmatrix}$$

n=3

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.5 & -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & -3k & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\mu & k\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2(\mu+\rho\sigma) & \sigma^2+2k\theta & -1 & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mu+2\rho\sigma & 3(\sigma^2+k\theta) & -0.5 & -2k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & k\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mu+\rho\sigma & \sigma^2+2k\theta & -0.5 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & k\theta & 0 \end{pmatrix}$$

$$\text{with corresponding } y(t)=\begin{pmatrix} C_{30} \\ C_{21} \\ C_{12} \\ C_{03} \\ C_{20} \\ C_{11} \\ C_{02} \\ C_{10} \\ C_{01} \\ C_{00} \end{pmatrix}$$

n=4

$$
A=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.5 & -2k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -3k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5 & -4k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4\mu & k\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 3(\mu+k\theta) & \sigma^2+2k\theta & 0 & 0 & -1.5 & -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 2\mu+4\rho\sigma & 3(\sigma^2+k\theta) & 0 & 0 & -1 & -2k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \mu+3\rho\sigma & 6\sigma^2+4k\theta & 0 & 0 & -0.5 & -3k & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3\mu & k\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 2(\mu+\rho\sigma) & \sigma^2+2k\theta & 0 & -1 & -k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu+2\rho\sigma & 3(\sigma^2+k\theta) & 0 & -0.5 & -2k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & k\theta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu+\rho\sigma & \sigma^2+2k\theta & -0.5 & -k & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & k\theta & 0
\end{pmatrix}
$$

with corresponding

$$
y(t)=\begin{pmatrix} C_{40} \\ C_{31} \\ C_{22} \\ C_{13} \\ C_{04} \\ C_{30} \\ C_{21} \\ C_{12} \\ C_{03} \\ C_{20} \\ C_{11} \\ C_{02} \\ C_{10} \\ C_{01} \\ C_{00} \end{pmatrix}
$$

The 1st to 3rd moments are:

$M(x,v,\tau,1)=$

$$
1.\,\mathrm{mu}\,\mathrm{tau}+\frac{\left(0.5-0.5\,e^{-1.\,k\,\mathrm{tau}}\right)\mathrm{theta}}{k}-0.5\,\mathrm{tau}\,\mathrm{theta}+\frac{\left(-0.5+0.5\,e^{-1.\,k\,\mathrm{tau}}\right)v}{k}+x
$$

$M(x,v,\tau,2)=$

$$-\frac{1.\ theta}{k}+\frac{1.\ e^{-1.\ k\ tau}\ theta}{k}+\frac{2.\ rho\ sigma\ theta}{k^2}-\frac{2.\ e^{-1.\ k\ tau}\ rho\ sigma\ theta}{k^2}-\frac{0.625\ sigma^2\ theta}{k^3}+$$

$$\frac{0.125\ e^{-2.\ k\ tau}\ sigma^2\ theta}{k^3}+\frac{0.5\ e^{-1.\ k\ tau}\ sigma^2\ theta}{k^3}+\frac{0.25\ theta^2}{k^2}+\frac{0.25\ e^{-2.\ k\ tau}\ theta^2}{k^2}-\frac{0.5\ e^{-1.\ k\ tau}\ theta^2}{k^2}+$$

$$tau^2\ (1.\ mu^2-1.\ mu\ theta+0.25\ theta^2)+\frac{1.\ v}{k}-\frac{1.\ e^{-1.\ k\ tau}\ v}{k}-\frac{1.\ rho\ sigma\ v}{k^2}+\frac{1.\ e^{-1.\ k\ tau}\ rho\ sigma\ v}{k^2}+\frac{0.25\ sigma^2\ v}{k^3}-$$

$$\frac{0.25\ e^{-2.\ k\ tau}\ sigma^2\ v}{k^3}-\frac{0.5\ theta\ v}{k^2}-\frac{0.5\ e^{-2.\ k\ tau}\ theta\ v}{k^2}+\frac{1.\ e^{-1.\ k\ tau}\ theta\ v}{k^2}+\frac{0.25\ v^2}{k^2}+\frac{0.25\ e^{-2.\ k\ tau}\ v^2}{k^2}-\frac{0.5\ e^{-1.\ k\ tau}\ v^2}{k^2}+$$

$$\frac{1.\ theta\ x}{k}-\frac{1.\ e^{-1.\ k\ tau}\ theta\ x}{k}-\frac{1.\ v\ x}{k}+\frac{1.\ e^{-1.\ k\ tau}\ v\ x}{k}+x^2+tau\ \Big(1.\ theta+\frac{1.\ mu\ theta}{k}-\frac{1.\ e^{-1.\ k\ tau}\ mu\ theta}{k}-\frac{1.\ rho\ sigma\ theta}{k}-$$

$$\frac{1.\ e^{-1.\ k\ tau}\ rho\ sigma\ theta}{k}+\frac{0.25\ sigma^2\ theta}{k^2}+\frac{0.5\ e^{-1.\ k\ tau}\ sigma^2\ theta}{k^2}-\frac{0.5\ theta^2}{k}+\frac{0.5\ e^{-1.\ k\ tau}\ theta^2}{k}-\frac{1.\ mu\ v}{k}+$$

$$\frac{1.\ e^{-1.\ k\ tau}\ mu\ v}{k}+\frac{1.\ e^{-1.\ k\ tau}\ rho\ sigma\ v}{k}-\frac{0.5\ e^{-1.\ k\ tau}\ sigma^2\ v}{k^2}+\frac{0.5\ theta\ v}{k}-\frac{0.5\ e^{-1.\ k\ tau}\ theta\ v}{k}+2.\ mu\ x-1.\ theta\ x\Big)$$

$M(x,v,\tau,3)=$

$$\frac{1}{k^5}\ e^{-6.\ k\ tau}\ \Big(e^{3.\ k\ tau}\ \big(sigma^4\ (-0.0625\ theta+0.1875\ v)+$$
$$k\ sigma^2\ (-0.1875\ theta^2+0.5625\ theta\ v-0.375\ v^2)+k^2\ (-0.125\ theta^3+0.375\ theta^2\ v-0.375\ theta\ v^2+0.125\ v^3)\big)+$$
$$e^{4.\ k\ tau}\ \big(sigma^4\ (-0.375\ theta+0.375\ v)+k\ sigma^2\ (-0.5625\ theta^2+rho\ sigma\ (1.5\ theta-2.25\ v)+$$
$$sigma^2\ tau\ (-0.375\ theta+0.75\ v)+0.1875\ theta\ v+0.375\ v^2)+k^2\ (0.375\ theta^3+rho\ sigma^3\ tau\ (0.75\ theta-1.5\ v)-$$
$$1.125\ theta^2\ v+1.125\ theta\ v^2-0.375\ v^3+rho\ sigma\ (3.\ theta^2-4.5\ theta\ v+1.5\ v^2)+$$
$$sigma^2\ (-0.9375\ tau\ theta^2+v\ (1.5-0.75\ mu\ tau-0.75\ tau\ v-0.75\ x)+theta\ (-0.75+0.375\ mu\ tau+1.875\ tau\ v+0.375\ x)))+$$
$$k^3\ (-0.375\ tau\ theta^3+theta\ v\ (3.-1.5\ mu\ tau-3.\ rho\ sigma\ tau-0.375\ tau\ v-1.5\ x)+$$
$$v^2\ (-1.5+0.75\ mu\ tau+1.5\ rho\ sigma\ tau+0.75\ x)+theta^2\ (-1.5+0.75\ mu\ tau+1.5\ rho\ sigma\ tau+0.75\ tau\ v+0.75\ x))\big)+$$
$$e^{5.\ k\ tau}\ \big(sigma^4\ (-0.9375\ theta-0.1875\ v)+k\ sigma^2\ (6.\ rho\ sigma\ theta+1.6875\ theta^2+sigma^2\ tau\ (-1.125\ theta+0.375\ v)-$$
$$2.0625\ theta\ v+0.375\ v^2)+k^2\ (-0.375\ theta^3+rho\ sigma^3\ tau\ (6.\ theta-3.\ v)+sigma^4\ tau^2\ (-0.375\ theta+0.375\ v)+$$
$$1.125\ theta^2\ v-1.125\ theta\ v^2+0.375\ v^3+rho\ sigma\ (-6.\ theta^2+9.\ theta\ v-3.\ v^2)+$$
$$sigma^2\ (-0.375\ tau\ theta^2+v\ (3.\ rho^2+0.75\ tau\ v)+theta\ (-3.-9.\ rho^2+1.5\ mu\ tau-1.125\ tau\ v+1.5\ x)))+$$
$$k^4\ (tau^2\ theta^2\ (-0.375\ theta+0.375\ v)+mu^2\ tau^2\ (-1.5\ theta+1.5\ v)+rho^2\ sigma^2\ tau^2\ (-1.5\ theta+1.5\ v)+$$
$$tau\ theta\ (v\ (3.-1.5\ x)+theta\ (-3.+1.5\ x))+x\ (theta\ (3.-1.5\ x)+v\ (-3.+1.5\ x))+$$
$$rho\ sigma\ tau\ (1.5\ tau\ theta^2+theta\ (3.-1.5\ tau\ v-3.\ x)+v\ (-3.+3.\ x))+$$
$$mu\ tau\ (1.5\ tau\ theta^2+theta\ (3.-3.\ rho\ sigma\ tau-1.5\ tau\ v-3.\ x)+v\ (-3.+3.\ rho\ sigma\ tau+3.\ x)))+$$
$$k^3\ (0.75\ tau\ theta^3+rho^2\ sigma^2\ tau\ (-6.\ theta+3.\ v)+v^2\ (3.-1.5\ mu\ tau-1.5\ x)+$$
$$theta^2\ (3.-1.5\ mu\ tau-1.5\ tau\ v-1.5\ x)+theta\ v\ (-6.+3.\ mu\ tau+0.75\ tau\ v+3.\ x)+$$
$$sigma^2\ tau\ (-0.75\ tau\ theta^2+v\ (3.-1.5\ mu\ tau-1.5\ x)+theta\ (-3.+1.5\ mu\ tau+0.75\ tau\ v+1.5\ x))+$$
$$rho\ sigma\ (3.\ tau\ theta^2+theta\ (6.-6.\ mu\ tau+1.5\ sigma^2\ tau^2-6.\ x)+v\ (-3.+tau\ (3.\ mu-1.5\ sigma^2\ tau-1.5\ v)+3.\ x)))\big)+$$
$$e^{6.\ k\ tau}\ \big(sigma^4\ (1.375\ theta-0.375\ v)+k\ sigma^2\ (-0.375\ sigma^2\ tau\ theta-0.9375\ theta^2+1.3125\ theta\ v-$$
$$0.375\ v^2+rho\ sigma\ (-7.5\ theta+2.25\ v))+$$
$$k^2\ (2.25\ rho\ sigma^3\ tau\ theta+0.125\ theta^3-0.375\ theta^2\ v+0.375\ theta\ v^2-0.125\ v^3+rho\ sigma\ (3.\ theta^2-4.5\ theta\ v+1.5\ v^2)+$$
$$sigma^2\ (1.3125\ tau\ theta^2+theta\ (3.75+9.\ rho^2-1.875\ mu\ tau-0.75\ tau\ v-1.875\ x)+v\ (-1.5-3.\ rho^2+0.75\ mu\ tau+0.75\ x)))+$$
$$k^5\ (1.\ mu^3\ tau^3-0.125\ tau^3\ theta^3+tau^2\ theta^2\ (-1.5+0.75\ x)+tau\ theta\ (3.-1.5\ x)\ x+x^3+$$
$$mu^2\ tau^2\ (-1.5\ tau\ theta+3.\ x)+mu\ tau\ (0.75\ tau^2\ theta^2+tau\ theta\ (3.-3.\ x)+3.\ x^2))+$$
$$k^4\ (mu^2\ tau^2\ (1.5\ theta-1.5\ v)+tau^2\ theta^2\ (0.375\ theta-0.375\ v)+rho\ sigma\ tau\ theta\ (3.+1.5\ tau\ theta-3.\ x)+$$
$$x\ (v\ (3.-1.5\ x)+theta\ (-3.+1.5\ x))+tau\ theta\ (theta\ (3.-1.5\ x)+v\ (-3.+1.5\ x))+$$
$$mu\ tau\ (-1.5\ tau\ theta^2+v\ (3.-3.\ x)+theta\ (-3.-3.\ rho\ sigma\ tau+1.5\ tau\ v+3.\ x)))+$$
$$k^3\ (-3.\ rho^2\ sigma^2\ tau\ theta-0.375\ tau\ theta^3+theta\ v\ (3.-1.5\ mu\ tau-0.375\ tau\ v-1.5\ x)+v^2\ (-1.5+0.75\ mu\ tau+0.75\ x)+$$
$$sigma^2\ tau\ theta\ (-1.5+0.75\ mu\ tau-0.375\ tau\ theta+0.75\ x)+theta^2\ (-1.5+0.75\ mu\ tau+0.75\ tau\ v+0.75\ x)+$$
$$rho\ sigma\ (-4.5\ tau\ theta^2+v\ (3.-3.\ mu\ tau-3.\ x)+theta\ (-6.+6.\ mu\ tau+3.\ tau\ v+6.\ x)))\big)\Big)$$